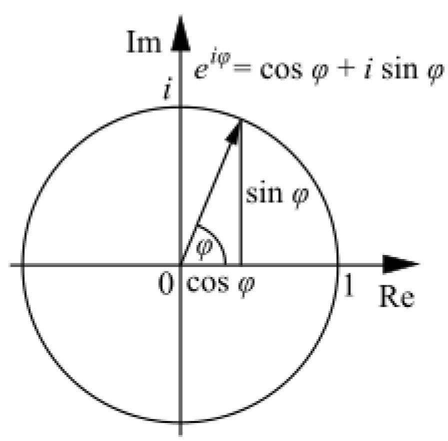
Tutorial on Quantum Theory

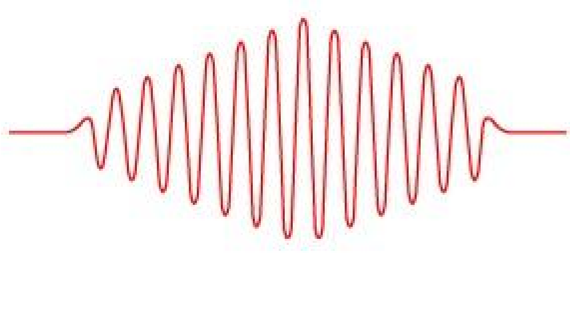
Ordinary quantum theory is about the motion of point particles. In classical physics we have the notion that a point particle follows some trajectory $x=x(t)$ defined in some coordinate system. The point x localizes the position of the particle in that coordinate system, and the particle moves with a momentum $p=mv$, where $v=dx/dt$. In quantum theory we define a wave-function for the particle $\psi(x,t)$, which specifies the quantum probability that the particle can be localized at position x at time t. The wave-function is a superposition of all possible position eigenstates, which by Fourier transform can be written as a superposition of all momentum eigenstates. A Fourier transform transforms the wave-function from position to momentum space as $\psi(x,t)\rightarrow\psi(p,t)$. A momentum eigenstate is defined by the eigenvalue equation $-i\hbar\partial\psi/\partial x=p\psi$, with solution $\psi=A\exp(ipx/\hbar)$, where $\hbar=h/2\pi$. This momentum eigenstate is understood as a plane wave of amplitude A, where A depends on the momentum p. If we impose periodic boundary conditions on the wave-function in a box of size 2L, $\psi(x=-L)=\psi(x=L)$, then we quantize momentum as $p=nh/2L$, where n is a quantum number that runs over the integers n=1,2,3,... With this periodic boundary condition, $px/\hbar=\pm\pi n$ at the boundaries $x=\pm L$, which is periodic in the sense that $\exp(\pm\pi in)$ represents $\pm\pi n$ rotations around the complex plane at the boundaries $x=\pm L$. This is easiest to see with Euler's equation, $\exp(i\theta)=\cos\theta+i\sin\theta$, which represents a rotation in the complex plane by an angle θ.



Complex Plane

The most general wave-function, which is a superposition of all possible momentum eigenstates, is written as $\psi(x,t)=\int dpA(p)\exp i(px-Et)/\hbar$, where for the motion of a free Newtonian particle $E=p^2/2m=\frac{1}{2}mv^2$ is the kinetic energy of the particle. The momentum plays the role of a wavelength $p=h/\lambda$ and the energy plays the role of a wave frequency $E=hf$. Notice that for the above example of a particle in a box, the condition of periodic boundary conditions on the wave-function specifies that n wavelengths have to fit into the length of the box as $n\lambda=2L$, which are called standing waves and is the basic mechanism in quantum theory by which momentum is quantized. The amplitude A(p) specifies the quantum probability with which the particle can be measured at a classical position $x=x_0$ at time $t=t_0$. The way this works is in terms of a wave-packet that is localized around $x=x_0$ at time $t=t_0$. In terms of a classical particle trajectory, $x_0=x(t=t_0)$. By Fourier transform, we can also write the wave-function in terms of a superposition of all possible position eigenstates, which are defined in terms of the eigenvalue equation $i\hbar\partial\psi/\partial p=x\psi$. These position eigenstates are defined in terms of the delta-function $\boldsymbol{\delta}(x-x_0)=N\int dp\exp(ip(x-x_0)/\hbar)$, which is highly peaked around $x=x_0$, and where N is a normalization factor that normalizes the total probability of measuring the particle at any possible position coordinate to P=1. In this way we can write the wave-function for the particle either in terms of a superposition of all possible momentum eigenstates or a superposition of all possible position eigenstates. The wave-function will typically be localized around some classical value of the momentum $p=p_0$ in momentum space and some classical value of position $x=x_0$ in position space. Notice that a position eigenstate is given by A(p)=1 and a momentum eigenstate by $A(p)=\boldsymbol{\delta}(p-p_0)=N\int dx\exp(i(p-p_0)x/\hbar)$, which reflects the nature of the Fourier transform. If we think of the canonical variables x and p as differential operators that operate on the wave-function, then with $p=-i\hbar\partial/\partial x$, these operators obey a commutation relation of the form $px-xp=-i\hbar$.

For the most general possible wave-function, formulated either as a superposition of all possible momentum eigenstates or a superposition of all possible position eigenstates, A(p) is something in between a highly peaked delta-function and a constant, which is the nature of the wave-packet. Since the wave-function written as either a superposition of all possible momentum eigenstates or a superposition of all possible position eigenstates is peaked in the sense of a wave-packet localized around the classical values of $p=p_0$ and $x=x_0$, we can specify an uncertainty principle for these uncertainties in the measurements of the position and the momentum of the particle as $\Delta x\Delta p\geq\hbar$. The uncertainty principle is inherent in how highly localized the wave-function is peaked as a wave-packet around the classical values of position and momentum. The more we know about the position of the particle, the less we know about its momentum, and vice versa.



Wave-packet

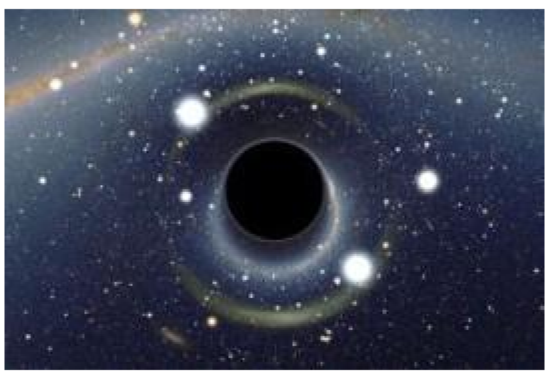
The uncertainty principle simply reflects that the wave-function is a probability amplitude that specifies the quantum probability with which the particle can be measured at a position x and to move with a momentum p at any time t. The more highly peaked the wave-function is as a localized wave-packet in position space, the less localized it will be in momentum space, and vice versa, as mathematically represented by a Fourier transform of the wave-function that transforms it from position to momentum space.

A measurement of the position of the particle reduces the wave-function written as a superposition of all possible position eigenstates to a particular position eigenstate, and a measurement of the momentum of the particle reduces the wave-function written as a superposition of all possible momentum eigenstates to a particular momentum eigenstate. This reduction of the quantum state to some particular measurable state of the particle is the essential nature of the measurement problem of quantum theory, which is either called a quantum state reduction or the collapse of the wave-function. With any measurement of the particle's position or momentum, the more we know about one of these variables, the less we know about the other variable.

Notice that the wave-function also obeys the eigenvalue equation $i\hbar\partial\psi/\partial t=E\psi$, This is the origin of the most general wave equation for the particle's wave-function, which takes the form $i\hbar\partial\psi/\partial t=H\psi$, where H is the Hamiltonian or total energy operator. This wave equation has the solution $\psi(t)=\exp(-iHt/\hbar)\psi(0)$, which is a statement of the unitary time evolution of the wave-function. This is usually taken to be the defining equation of quantum theory, which directly leads to the Feynman sum over all possible paths formulation of quantum theory and to the Feynman diagram formulation of quantum field theory. For a non-relativistic Newtonian particle, total energy $E=p^2/2m+V(x)$, which generates the Schrodinger wave equation with $p=-i\hbar\partial/\partial x$ as part of the Hamiltonian operator operating on the wave-function. We again have an eigenvalue equation for the wave-function $H\psi(x,t)=E\psi(x,t)$, which gives quantized values for energy when periodic boundary conditions are imposed on the wave-function. The situation for relativistic particles is more complicated. For the relativistic spin ½ electron, the wave equation is specified by Dirac's equation for the spinor electron field, and for the massless spin 1 photon, the wave equation is specified by Maxwell's equations for the electromagnetic field. The reason relativistic particles are different from classical Newtonian particles is because the motion of a free relativistic particle obeys $E^2=p^2c^2+m^2c^4$, which in terms of energy and momentum operators that operate on a wave-function generates the Klein-Gordon equation. For the massless spin 1 photon, the energy-momentum relation is simply $E=pc$, which implies $f=c/\lambda$. The relativity that is at work here is special relativity, which is the relativity of flat Minkowski space, within which there is no effect of gravity.

The situation with gravity is again more complicated. By its very nature, gravity is described by a dynamically curved space-time geometry, which is formulated in terms of Einstein's field equations for the space-time metric. In the sense of quantum field theory, which describes the motion of point particles, Einstein's field equations are the wave equations for the massless spin 2 graviton. However, there are a bunch of big problems when we try to conceptualize how we would measure the location of the graviton as a point particle in some coordinate system. The first problem is defining that coordinate system. Quantum field theory as a formulation of the motion of point particles in a coordinate system is really only defined in the flat gravity-free Minkowski space of special relativity. By its nature, gravity is only described in terms of a dynamically curved space-time geometry, and to consider the graviton as a point particle propagating through flat gravity-free Minkowski space is a logical contradiction. The second problem has to do with the nature of black holes and what can actually be measured.

It turns out that it's logically impossible to measure the position of the graviton as a point particle in some coordinate system due to the nature of black holes. This has to do with the problem of a smallest possible distance scale that can be measured, called the Planck length, which arises when the general relativity of gravity is combined with basic notions of quantum theory. It's actually quite easy to calculate this smallest possible measurable distance scale using the basic idea of the event horizon of a black hole.



Black Hole

A black hole is characterized by an event horizon, which is a boundary in space beyond which nothing is observable. At the event horizon of a black hole, escape velocity is the speed of light, and since nothing can travel faster than the speed of light in three dimensional space, nothing is observable beyond the limits of the two dimensional boundary of the event horizon, where the force of gravity is so strong that even light cannot escape from the black hole. It's easy to calculate the radius of the event horizon using ordinary concepts of classical physics. Escape velocity is defined when a particle has just enough kinetic energy that it can overcome the attractive force of gravity. A classical particle of mass m that moves with a velocity v and is located at a radius R from the central point of a larger mass M that generates a gravitational field has a total energy that's given in terms of these parameters as $E=\frac{1}{2}mv^2-GMm/R$. The minus sign indicates that gravity is an attractive force. With escape velocity, the particle has just enough kinetic energy of motion to overcome this gravitational attraction, and eventually escapes away to infinity, at which point it stops moving. This specifies the particle's escape velocity in terms of a zero total energy, $E=0$, which gives it an escape velocity of $v^2=2GM/R$. The event horizon of a black hole is a two dimensional surface of space surrounding the mass M within which nothing can escape away from the black hole, which means escape velocity at the event horizon is the speed of light, $v=c$. This gives the radius of the event horizon in terms of the mass of the black hole as $R=2GM/c^2$.

$$R = \frac{2GM}{c^2}$$

Schwarzschild Radius of a Black Hole

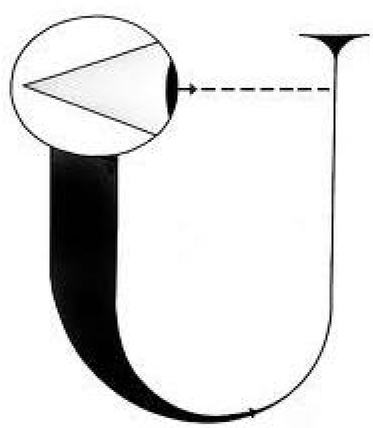
With the radius of the event horizon of a black hole and some basic notions of quantum theory, we can now calculate the smallest possible measurable distance scale, which is called the Planck length. The way we measure the size of an object is by scattering light off the object. To measure the size of a smaller object, we have to use light that has a smaller wavelength. We can use visible light in an ordinary microscope to measure the size of a bacterium, but to measure the size of a virus, we have to use x-rays in an electron microscope, which have a smaller wavelength. When we measure the size of an object, we are literally scattering photons off the object. In quantum theory, the energy of a photon is related to the frequency of its wave vibrations, $E=hf$, and since the frequency is given in terms of the speed of light and its wavelength as $f=c/\lambda$, the energy of the photon is given in terms of its wavelength as $E=hf=hc/\lambda$. This means to measure the size of a smaller object, we have to use light that has a smaller wavelength and a higher energy. As we measure smaller and smaller objects, we eventually concentrate so much energy into such a small region of space that we create a black hole. The mass of the black hole is determined in terms of the energy of the photon that we have to use to measure the size of the object, $E=Mc^2=hc/\lambda$. The distance scale at which the black hole forms is called the Planck length, ℓ. If we set this distance scale equal to both the radius of the event horizon of the black hole and the wavelength of the photon that is scattered off the object as $\ell=R=\lambda$, and use $R=2GM/c^2$, then $\ell=R=2GM/c^2=2hG/\lambda c^3$, and with $\ell=\lambda$, we arrive at the final result that $\ell=2hG/\ell c^3$, which is the distance scale at which the black hole must form. The Planck length is defined as $\ell^2=\hbar G/c^3$.

$$\ell_p = \sqrt{\frac{\hbar G}{c^3}} \sim 1.6 \times 10^{-35}\text{m}$$

Planck Length

There is an ultimate distance scale that we can measure, called the Planck length. If we try to measure an object smaller than a Planck length, we have to concentrate so much energy into such a small region of space that we create a black hole, and then we can measure nothing beyond the limits of its event horizon. This is why we cannot measure a graviton as a point particle. If we try to measure a graviton as a point particle, we only end up creating a Planck-size black hole, and then we can measure nothing beyond its event horizon. It seems relativity theory forbids the measurement of a graviton as a point particle, which is another way of saying that there is no such thing as a point particle or graviton formulation of gravity. Gravity cannot be understood as a quantum field theory since that is inherently a point particle formulation. It is impossible to quantize Einstein's field equations for the space-time metric as a quantum field theory.

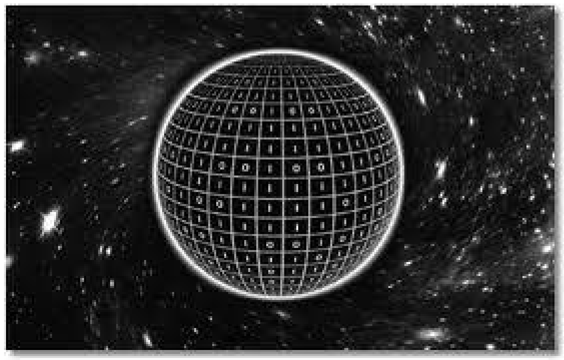
What is the solution for this problem? The answer of course is the holographic principle. Gravity can only be quantized when an observer, which is the perceiving consciousness present at the central point of view of its own holographic world, enters into an accelerated frame of reference, within which its event horizon arises that becomes the observer's holographic screen when its horizon encodes qubits of information.



Universal Observer

The holographic principle tells us that all the qubits of information that give rise to the gravitational field of a black hole are encoded on the event horizon of the black hole.

The smallest possible black hole is a Planck-size black hole that encodes a single qubit of information. This explains how information is encoded in quantum gravity. A Planck area defined on an accelerating observer's event horizon is the quantum of space-time geometry, and a qubit is the quantum of information.



Black Hole Information